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On the Fitting of Three-Phase Voltage Curves with Ellipses Using Geometric Algebra for Conics

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Abstract. This work introduces a novel approach in power quality analysis by fitting three-phase voltage curves with ellipses using Geometric Algebra for Conics (GAC). Recognizing the limitations of current linear algebra-based methods, this paper highlights the need for further exploration into more effective techniques. Geometric Algebra offers a potent, efficient, and intuitive alternative for addressing power quality phenomena, such as sags, swells, and interruptions. By establishing a robust mathematical foundation with GAC and analysing the characteristics of the resulting ellipse, we propose new metrics for understanding power quality issues. This foundational paper lays the groundwork for future, more realistic studies, inviting the engineering community to explore the benefits of GA in advancing power quality analysis. Through this initial exploration, we aim to spark further research and development in this promising field.

Key words. Geometric algebra for conics, power quality, ellipse fitting, geometric electricity.

1. Introduction

The analysis of power quality is a critical concern in electrical engineering, with phenomena like voltage sags, swells, and interruptions having significant impacts on system performance and equipment lifespan. Traditionally, linear algebra has provided the mathematical foundation for characterizing and studying power quality issues [1]. However, strong limitations are widely known in using linear techniques for fitting multi-phase voltage curves (typically known as space vectors) and extracting insightful parameters [2,3,4]. This paper proposes an alternative approach using Geometric Algebra for Conics (GAC) to model three-phase voltages spatial curves as ellipses.

Geometric Algebra provides a powerful mathematical framework unifying vectors, complex numbers, quaternions, and matrices into a coherent language [5]. GAC offers an intuitive and efficient toolset for conic analysis, with ellipses constituting one of the fundamental conic primitives [6]. By leveraging GAC, voltage curves can be elegantly encapsulated as ellipses, providing direct access to insightful geometric and algebraic properties. These include the ellipse centre, axes, eccentricity, and rotational orientation, enabling new perspectives and quantifiable metrics for power quality studies.

While initial GAC research has shown promising results [7], its application to engineering problems, including modelling voltage curves and power quality, remains largely unexplored. This paper aims to establish the theoretical foundations and demonstrate the feasibility of using GAC-based ellipses for multi-phase voltage characterization. We present the methodology for fitting ellipses to empirical voltage data, including computational techniques and geometrical interpretations. Both averaged and instantaneous approaches are considered, accommodating different data sources and applications.

Through numerical experiments on simulated voltage curves, we showcase the viability of ellipse fitting with GAC. The preliminary results exhibit GAC's benefits in model accuracy, parameter extraction, and geometric intuitiveness. By introducing GAC to the power quality research community through this foundational study, we hope to spur further investigations into its capabilities for voltage analysis and beyond. The long-term goal is to develop a comprehensive GA-based framework for electrical engineering, unlocking new physical insights and mathematical perspectives.

2. Mathematical Background

A. Basics of Geometric Algebra

Geometric algebra is a powerful mathematical framework that extends the concepts of linear algebra, vectors, complex numbers, and matrices to a new level of abstraction and generality. It provides a unified language for engineering, enabling the concise description of geometric transformations, complex rotations, and the modelling of physical systems in general, and power systems in particular [8,9,10,11]. At the heart of geometric algebra is the geometric product, an operation that combines aspects of the dot and cross products familiar from vector algebra.
The geometric product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined as:

$$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$$

where $\boldsymbol{a} \cdot \boldsymbol{b}$ is the dot product, representing the scalar or magnitude aspect of the multiplication, and $\boldsymbol{a} \wedge \boldsymbol{b}$ is the wedge product, representing the oriented area spanned by the two vectors (known as bivector), akin to the vector product in three dimensions but extended to higher dimensions.
This compact and elegant operation encapsulates both the magnitude and direction information, enabling a straightforward representation of complex geometric concepts such as rotations via rotors (exponential of bivectors), reflections, and projections in a multidimensional space. Geometric algebra's expressive power lies in its ability to simplify equations and algorithms in geometry, physics, and engineering, making it an invaluable tool for both theoretical exploration and practical application.

B. Geometric Algebra for Conics

Geometric algebra offers a robust framework for analyzing 2D spaces, particularly useful for working with conic sections. Central to this approach is the Compass Ruler Algebra (CRA), a Clifford algebra denoted as $\mathcal{C}\ell(3,1)$. This algebra enables the representation of geometric objects in a 2D plane through an affine subalgebra generated by vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4$ where $\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = \boldsymbol{e}_3^2 = 1$ and $\boldsymbol{e}_4^2 = -1$. For mathematical convenience, it is useful to work with two additional null vectors (Witt pairs) $\boldsymbol{e}_0 = \frac{1}{2}(\boldsymbol{e}_3 - \boldsymbol{e}_4)$ and $\boldsymbol{e}_\infty = \boldsymbol{e}_3 + \boldsymbol{e}_4$ where $\boldsymbol{e}_0^2 = \boldsymbol{e}_\infty^2 = 0 \; and \; \boldsymbol{e}_0 \cdot \boldsymbol{e}_\infty = -1$. Thus, a point $P = (x, y)$ in $\mathbb{R}^2$ can be mapped as

$$\boldsymbol{P} \longrightarrow \boldsymbol{p} = \boldsymbol{e}_0 + \mathbf{x} + \frac{1}{2}\mathbf{x}^2 \boldsymbol{e}_\infty$$

with $\mathbf{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$. We can use the scalar product to represent a geometric object V in the following way

$$\mathrm{P} \in V \Leftrightarrow \boldsymbol{p} \cdot \boldsymbol{v} = 0$$

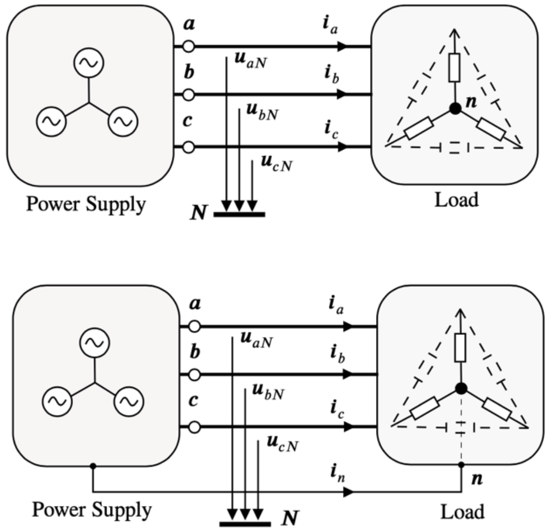

Fig. 1: Three-wire (top) and four-wire (bottom) three-phase power systems.

with $\boldsymbol{v} = v_0\boldsymbol{e}_0 + v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_\infty\boldsymbol{e}_\infty$. The element V can represent geometric objects based on monoms $1, x, y$ and $x^2 + y^2$. For example, a line

$$v_\infty + xv_1 + yv_2 = 0$$

can be obtained if $v_0 = 0$. Moreover, circles are also obtained if $v_0 < 0$. For our GAC, we need three copies of the Witt pairs: $\{\boldsymbol{e}_0, \boldsymbol{e}_\infty\}$, $\{\tilde{\boldsymbol{e}}_0, \tilde{\boldsymbol{e}}_\infty\}$ and $\{\bar{\boldsymbol{e}}_0, \bar{\boldsymbol{e}}_\infty\}$. This algebra allows us to linearize up to three monoms in $\mathbb{R}^2$, updating the previous map to

$$P \longrightarrow \boldsymbol{p} = \boldsymbol{e}_0 + \mathbf{x} + \frac{1}{2}\mathbf{x}^2\boldsymbol{e}_\infty + \frac{1}{2}(x^2 - y^2)\tilde{\boldsymbol{e}}_\infty + xy\bar{\boldsymbol{e}}_\infty$$

Thus, the inner product of the vector representation of an object V in GAC with an embedded point P is given by

$$\begin{aligned} \boldsymbol{p} \cdot \boldsymbol{v} \quad &= -\frac{1}{2}(v_+ + \tilde{v}_0)x^2 - \frac{1}{2}(v_+ - \tilde{v}_0)y^2 \\ &-\bar{v}_0 xy + xv_1 + yv_2 + v_0 \end{aligned}$$

and it represents an arbitrary conic section.

3. Problem Statement and Proposed Method

Multi-phase electrical systems can be analysed geometrically by means of vectors. The components of these vectors correspond to the values measured in each phase of the system. In the study of power quality, voltage is typically used as the appropriate measure. For three-phase systems, there are two cases depending on whether there is a neutral wire or not (see Fig. 1). Depending on the configuration and chosen measurement points, several voltage sets can be defined

- Line to virtual neutral voltage u_{iN}, where $i = a, b, c$ and N is the virtual neutral point.
- Line-Line voltage u_{ij}, with $i, j = a, b, c$ and $i \neq j$
- Line to neutral voltage u_{in}, where $i = a, b, c$ and n is the neutral wire.

Therefore, it is possible to construct at least three distinct vectors, which are commonly used in electrical engineering

$$\boldsymbol{u}_{LN} = u_{aN}\boldsymbol{e}_1 + u_{bN}\boldsymbol{e}_2 + u_{cN}\boldsymbol{e}_3$$
$$\boldsymbol{u}_{LL} = u_{ab}\boldsymbol{e}_1 + u_{bc}\boldsymbol{e}_2 + u_{ca}\boldsymbol{e}_3$$
$$\boldsymbol{u}_{Ln} = u_{an}\boldsymbol{e}_1 + u_{bn}\boldsymbol{e}_2 + u_{cn}\boldsymbol{e}_3$$

For three-wire systems, only the line-to-virtual-neutral and line-to-line voltages are considered. Defining the all-ones vector (Kirchhoff vector) $\boldsymbol{k} = \sum \boldsymbol{e}_i$, and since Kirchhoff's laws must be fulfilled, we have that

$$\boldsymbol{u}_{LN} \cdot \boldsymbol{k} = 0$$
$$\boldsymbol{u}_{LL} \cdot \boldsymbol{k} = 0$$

Note that the trajectory lies in the plane defined by normal vector $\boldsymbol{k}$.
For the case of four-wire systems, there's an additional wire, the neutral wire[1]. The variable commonly considered here is the line-to-neutral voltage. Applying Kirchhoff's laws, for the general case we now get $\boldsymbol{u}_{Ln} \cdot \boldsymbol{k} \neq 0$. This, in turn, implies that generally, trajectories aren't restricted to a plane.

To generate the spatial curves, the measurement system acquires samples at regular intervals based on the meter's sampling frequency. If the system is balanced and purely sinusoidal, with the same amplitude and a phase difference of 120º between phases, the curve will be a circle. However, if the system is unbalanced and sinusoidal, the result will be an ellipse. For non-sinusoidal systems, the resulting curve is flat and may change shape based on the amount of harmonics and voltage unbalance. The plane containing the curve remains consistent in all scenarios and corresponds to the dual of Kirchhoff vector $\boldsymbol{k}$. Consequently, our interest lies in identifying the ellipses that best fit the set of points defining the flat trajectory. This process can be approached from two perspectives: averaged or instantaneous.

- **Averaged**. In this context, we have a large set of points during a period of the signal. Depending on the sampling rate, this could be hundreds to thousands of points. Several techniques can be applied to reduce the number of points under consideration.
- **Instantaneous**. Here, we continually receive consecutive samples (potentially 10-20 samples) and we construct the best fitting ellipse. As new samples arrive, we adjust and modify the ellipse accordingly.

[1]Ffour-wire three-phase systems are the subject of discussion since they can be viewed as highly asymmetrical and unbalanced 4-phase systems.

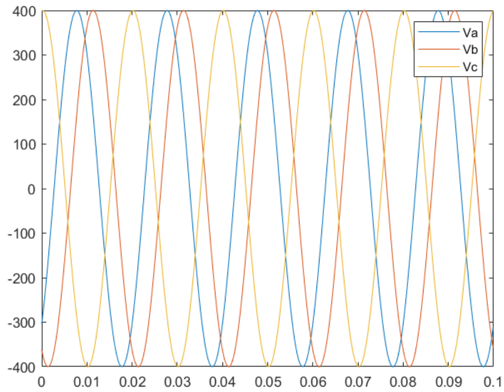

Fig. 2: Highly unbalanced sinusoidal voltages in a three-phase four-wire system for scenario 1.

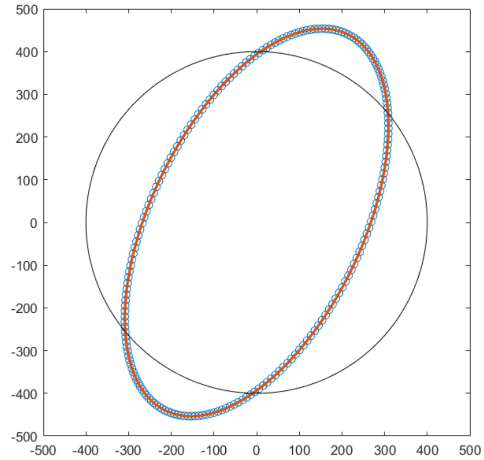

Fig. 3: Ellipse fitting for voltages in scenario 1.

4. Examples

In this section, some specific examples are presented to illustrate the proposed method. For this purpose, a series of synthetic three-phase voltage signals have been generated for a four-wire system with amplitude A = 400 V and frequency f = 50 Hz. Each example has been simulated for 5 cycles with a sampling frequency of 10000 Hz and a phase offset $\varphi = 2\pi/3$. Additionally, an amount of unbalance ϕ has been added to all signals. The expressions listed below are common to all examples:

$$w = 2\pi f$$
$$T = 1/f$$
$$-\pi/3 \leq \phi \leq \pi/3$$
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

A. Scenario 1: Unbalanced Sinusoidal Voltage

The initial example is depicted in a scenario with highly unbalanced sinusoidal voltages. The equations used to generate these signals are as follows:

$$v_a(t) = A\sin{(wt + \phi_a)}$$
$$v_b(t) = A\sin{(wt - \varphi + \phi_b)}$$
$$v_c(t) = A\sin{(wt - 2\varphi + \phi_c)}$$

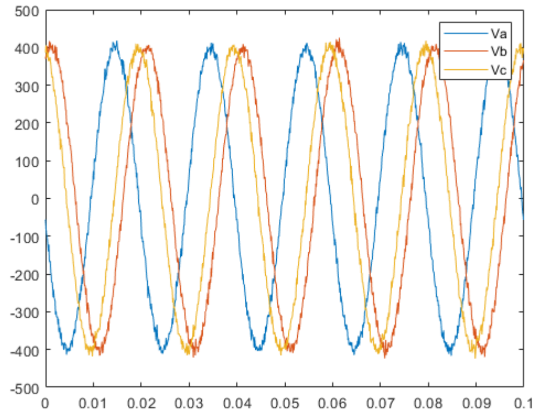

Fig. 4: Unbalanced sinusoidal signal with gaussian noise

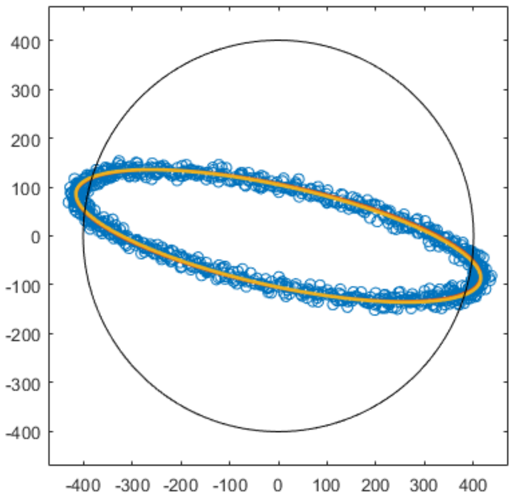

Fig. 5: Fitted ellipse for scenario 2.

These signals are represented in Fig. 2 and the fitted ellipse in Fig. 3. The black circle represents a perfect balanced and sinusoidal system with nominal amplitude.

B. Scenario 2: Unbalanced Voltage with Gaussian Noise.

This example shows the same unbalanced three-phase sine wave signal as in the previous example where Gaussian noise has been added at a ratio between 15 and 30 dB. Fig 4 and Fig. 5 show the signals and the fitted ellipse along with the nominal circle.

C. Scenario 3: Unbalanced Sinusoidal Voltage with Sag

In this scenario, an unbalanced sag is introduced. The equations used in this example are:

$$v_a(t) = A\left(1 - \alpha\big(u(t-t_1) - u(t-t_2)\big)\right) sin\,(wt + \phi_a)$$
$$v_b(t) = A\left(1 - \alpha\big(u(t-t_1) - u(t-t_2)\big)\right) sin\,(wt - \varphi + \phi_b)$$
$$v_c(t) = A\left(1 - \alpha\big(u(t-t_1) - u(t-t_2)\big)\right) sin\,(wt - 2\varphi + \phi_c)$$

where:

$$0.1 \leq \alpha \leq 0.9$$
$$T \leq t_2 - t_1 \leq (N-1)T$$

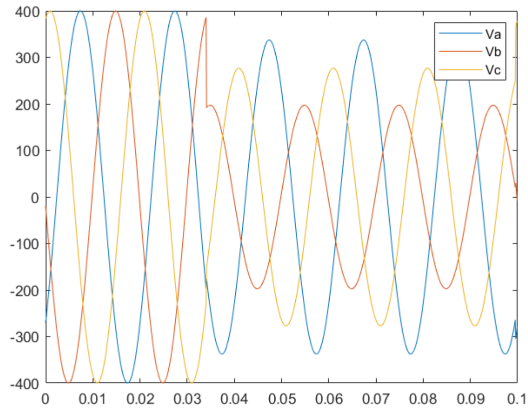

Fig. 6: Signals with unbalanced sag in scenario 3.

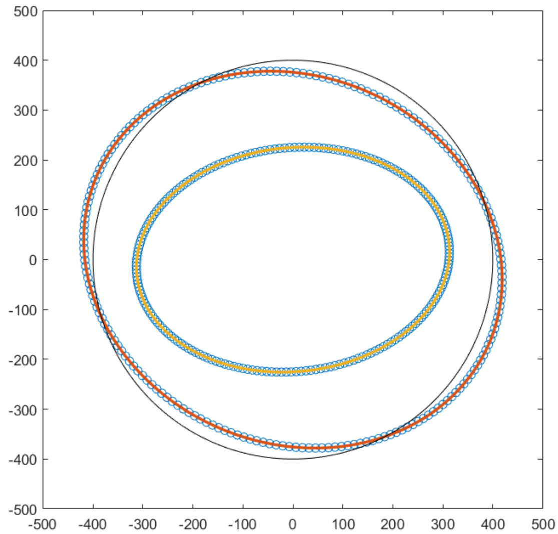

Fig. 7: Two ellipses are fitted for scenario 3, one for prefault and other during the sag.

Figs. 6 and 7 show the signals and the fitted ellipses. In this case, two different ellipses can be seen as a result of the prefault and fault condition. The proposed method can fit almost perfectly using the provided samples.

D. Scenario 4: Unbalanced Sinusoidal Voltage with Swell

For this example, the involved signals are

$$v_a(t) = A\left(1 + \beta\big(u(t-t_1) - u(t-t_2)\big)\right) \sin\,(wt + \phi_a)$$
$$v_b(t) = A\left(1 + \beta\big(u(t-t_1) - u(t-t_2)\big)\right) \sin\,(wt - \varphi + \phi_b)$$
$$v_c(t) = A\left(1 + \beta\big(u(t-t_1) - u(t-t_2)\big)\right) \sin\,(wt - 2\varphi + \phi_c)$$

where

$$0.1 \leq \beta \leq 0.8$$

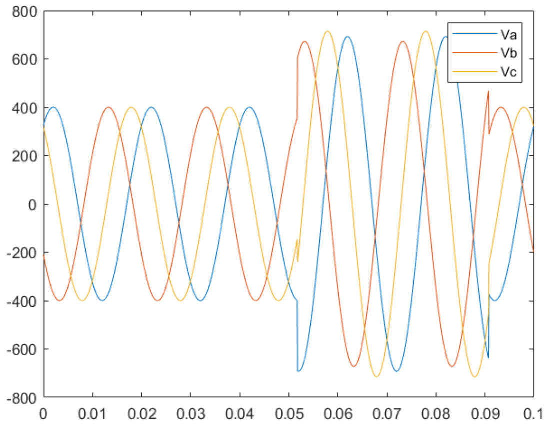

Fig. 8: Signals with unbalanced swell in scenario 4.

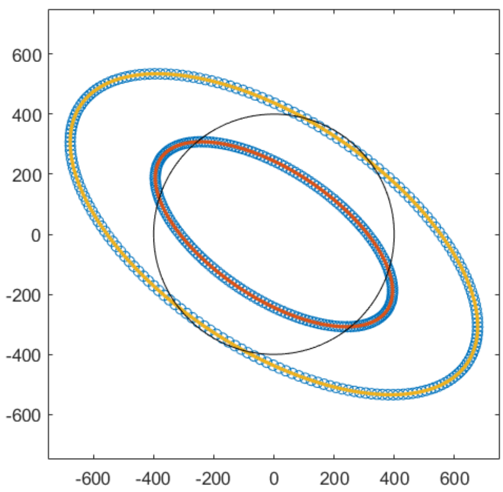

Fig. 9: Two ellipses are fitted for scenario 3, one for prefault and other during the swell.

E. *Scenario 5: Unbalanced Sinusoidal Voltage with Flicker*

For the flicker-type disturbance, the defining expressions are:

$$v_a(t) = A\left[1 + \lambda \sin\left(w_f t\right) - \alpha\left(u(t - t_1) - u(t - t_2)\right)\right]\sin\left(wt + \phi_a\right)$$
$$v_b(t) = A\left[1 + \lambda \sin\left(w_f t\right) - \alpha\left(u(t - t_1) - u(t - t_2)\right)\right]\sin\left(wt - \varphi + \phi_b\right)$$
$$v_c(t) = A\left[1 + \lambda \sin\left(w_f t\right) - \alpha\left(u(t - t_1) - u(t - t_2)\right)\right]\sin\left(wt - 2\varphi + \phi_c\right)$$

Where

$$0.05 \leq \lambda \leq 0.1$$
$$8 \leq f_f \leq 25Hz;\ w_f = 2\pi f_f$$

In this case, the ellipse fitting the flicker condition (Fig. 10) represents the least biased geometric ellipse (see Fig. 11).

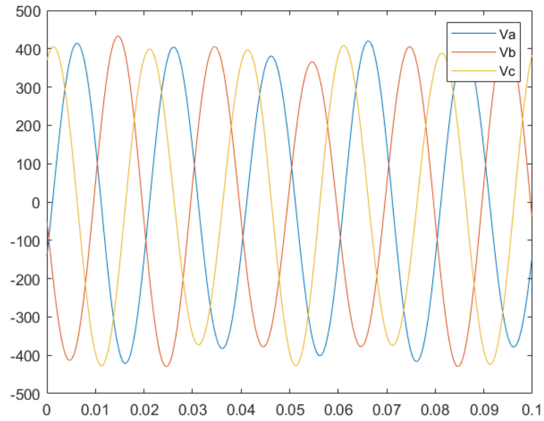

Fig. 10: Signals with flicker in scenario 5.

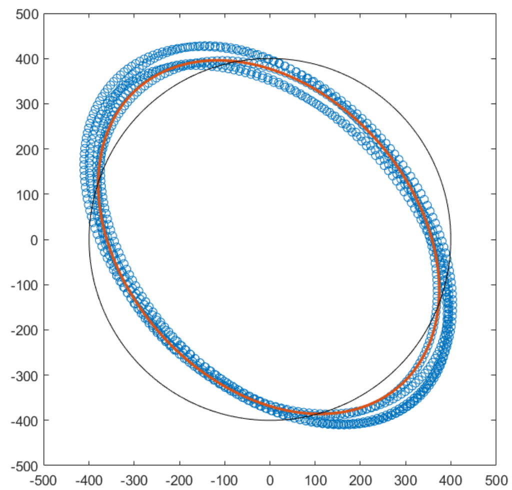

Fig. 11: Ellipse fitting for scenario 5 with flicker.

F. *Scenario 6: Unbalanced Voltage with Harmonics*

Finally, the voltage signals with harmonics are defined as

$$v_a(t) = A\left[\sin(wt + \phi_a) + \sum_{n=3}^{7} \alpha_{na}\sin(nwt - \vartheta_{na})\right]$$
$$v_b(t) = A\left[\sin(wt - \varphi + \phi_b) + \sum_{n=3}^{7} \alpha_{nb}\sin(nwt - \vartheta_{nb})\right]$$
$$v_c(t) = A\left[\sin(wt - 2\varphi + \phi_c) + \sum_{n=3}^{7} \alpha_{nc}\sin(nwt - \vartheta_{nc})\right]$$

with

$$n = \{3,5,7\}$$
$$0.05 \leq \alpha_n \leq 0.15$$
$$-\pi \leq \vartheta_n \leq \pi$$

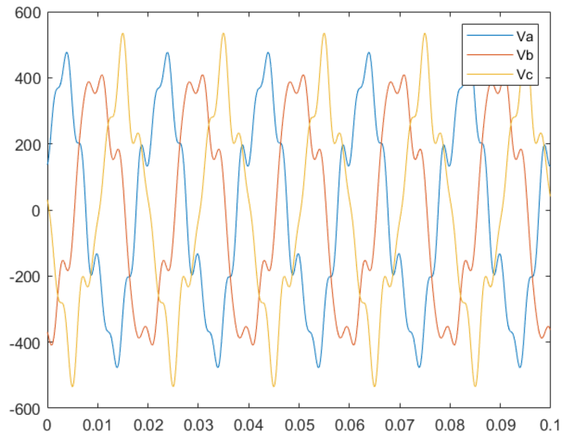

Fig. 12: Signals with unbalance harmonics in scenario 6.

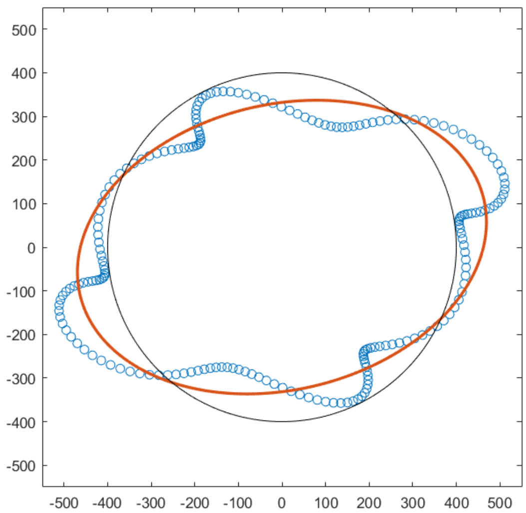

Fig. 13: Fitted ellipse for scenario 6.

Figs. 12 and 13 shows the signals and fitted ellipse

5. Discussion

Six different types of three-phase signals were synthetically generated for a four-wire system, to which ellipse fitting was applied using GAC. For each case, a plot of the three voltage signals and a plot for the ellipse fitting are shown. The circumference corresponding to a balanced three-phase sine wave signal is shown in black, for reference. The algorithm successfully identifies the ellipse that best resembles the signal in cases A, D, and E where effectively represents the mean of the signal path. In the case of the sag (example B) and the swell (example C), two ellipses are observed: one corresponding to the normal state of the system and the other due to the perturbation.

6. Conclusion

This work presents a novel method for power quality analysis using Geometric Algebra for Conics (GAC) to fit three-phase voltage curves as ellipses. The paper demonstrates how GAC provides an efficient framework for modelling voltages geometrically for further extraction of parameters such as centre, axes, orientation, and eccentricity.

The paper demonstrates the viability of GAC-based ellipse fitting through theoretical foundations and initial experiments on simulated data. This approach offers new insights into phenomena such as voltage sags and swells that affect power quality. The use of GAC for voltage modelling and analysis is pioneering, and this paper lays the groundwork for further research on extending GAC to related power system topics. Overall, GAC shows promise as a valuable new toolset for advancing electrical engineering applications.

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